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AVERAGE ELASTIC–PLASTIC BEHAVIOR OF COMPOSITE MATERIALS

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Abstract—A new approach to estimate the overall behavior of an inhomogeneous body is applied to investigate the average elastic–plastic state of composite materials in small deformation. The method is based on the concept of the average field of an infinite body with inhomogeneities, which is replaced by an equivalent body with homogeneous inclusions having appropriate eigenstrains that are composed of the actual plastic misfit strains and the properly determined eigenstrains by the equivalent replacement of the inhomogeneities. A simple example is given for a pure shear state of the elastic–perfectly plastic composite materials with spherical inhomogeneities. The overall hardening rate, quantitative estimate of ductility improvement, and unloading behavior are discussed.

1. INTRODUCTION

Attempting to understand the overall mechanical behavior of composite materials has been one of the major subjects in many engineering fields for many years. When the volume fraction of inhomogeneities is small, Eshelby's method evaluates such average behavior quite reasonably and explicitly including the shape effect of dispersoids (Eshelby, 1957).

However, when the interaction between inclusions becomes significant as the volume fraction increases, the so-called self-consistent method has been introduced by Kröner (1958), Budiansky (1965), and Hill (1965). This method can apply even for crystalline materials as a limiting case when the matrix portion of composites vanishes. Hutchinson (1964, 1970) has utilized this method to estimate the elastic–plastic incremental relations of crystalline materials and composites. Iwakuma and Nemat-Nasser (1984) and Nemat-Nasser and Iwakuma (1985) have extended the same method for finite deformations to predict ductile instability of strain localization. The method had been generalized and modified by Willis and Acton (1976), and Willis (1977). However, the self-consistent method fails to give a satisfactory estimate when the dispersoids are either voids or perfectly rigid, and yields unacceptable results when the volume fraction exceeds a certain value.

Recently several researches have been conducted to evaluate the average elastic moduli of composite materials. As far as the average field is concerned, the formulation becomes very simple by the Mori–Tanaka theory (Mori and Tanaka, 1973). Using the modification of the Mori–Tanaka method, Benveniste (1987) and Mori and Wakashima (1989) have estimated the average elastic moduli of composite materials. Here the aspect of a back stress is introduced to take into account the interaction of inhomogeneities and matrix materials.

Moreover the results become consistently identical in both cases where either the far-field stress or far-field displacement is prescribed. In other words, the estimation of material properties can be carried out independently of the far-field boundary conditions. This is strongly desirable from a physical point of view. The prediction by this method seems to give reasonable results even when the volume fraction becomes large. When the shape of all inhomogeneities with the same elasticity is spherical, it can be shown that the overall elastic moduli coincide with either one of the bounds calculated through a variational method by Hashin and Shtrikman (1963).

Here we apply this method to evaluate average elastic–plastic relations of composite materials within the framework of small deformations. For simplicity, a simple elastic-perfectly-plastic material is considered for the matrix and inhomogeneities. One of the main objectives is to calculate the average hardening coefficient of such two-phase composites.

2. DETERMINATION OF APPROXIMATE ELASTIC-PLASTIC BEHAVIOR

Consider a sufficiently large body D which contains identically shaped and randomly distributed and oriented inhomogeneities, $\Omega = \Omega_1 + \Omega_2 + \Omega_3 + \cdots$, the total volume fraction of which is denoted by f. The elastic moduli of the matrix M $(= D - \Omega)$ and inhomogeneities are $\tilde{C}$ and $\tilde{C}^*$, respectively. These materials are elastic-perfectly-plastic with the yield shear stresses τ_Y^M for the matrix and τ_Y^Ω for the inhomogeneity.

The following method based on that of Mori and Tanaka, yields the same results for the cases when either the surface traction or the surface displacement is prescribed on ∂D. As a matter of fact, this method has a great advantage because no boundary condition is necessary to determine the overall state. However, for the sake of clear comprehension, we here consider the case when the surface traction $(\tilde{n}\tilde{\sigma}^O)$ is applied on the boundary of the body whose outward unit normal vector is denoted by $\tilde{n}$. The case when the surface displacement is prescribed is given in the Appendix. The inhomogeneities are distributed randomly and are of the same shape. For the sake of simplicity, we consider only one representative volume V which contains only one inhomogeneity. Let the shape of the representative volume be similar to that of the inhomogeneity.

If no inhomogeneity exists, the overall strain $\tilde{\varepsilon}^O$ distributes uniformly, where $\tilde{\sigma}^O = \tilde{C}(\tilde{\varepsilon}^O - \tilde{\varepsilon}^P)$ and $\tilde{\varepsilon}^P$ denotes the overall uniform plastic strain. The existence of inhomogeneities provides disturbance in local fields of both the matrix and inhomogeneities. Let $\tilde{\sigma}_M$ and $\tilde{\sigma}_\Omega$ denote the average disturbed stress fields of the matrix and inhomogeneity, respectively. Then, since the overall stress field is prescribed, we must have

$$(1-f)\tilde{\sigma}_M + f\tilde{\sigma}_\Omega = 0. \tag{1}$$

Mori and Tanaka (1973) have shown that the average disturbed strain field due to the existence of an inhomogeneity vanishes outside of the inhomogeneity, provided that the shapes of the representative volume V and inhomogeneity Ω_K are taken as similar. Hence the average disturbed strain field due to the inhomogeneity exists only inside of the inclusion in the representative volume. However, the overall strain disturbance is no longer zero even outside of the inclusion because the surface displacement is free in the very far field. Therefore, the average field in the matrix can be written as

$$\tilde{\sigma}^O + \tilde{\sigma}_M = \tilde{C}(\tilde{\varepsilon}^O + \tilde{\varepsilon}_D - \tilde{\varepsilon}_M^P), \tag{2}$$

where $\tilde{\varepsilon}_M^P$ is the average plastic strain in the matrix, and $\tilde{\varepsilon}_D$ is the overall uniform disturbance. In other words, $\tilde{\varepsilon}_D$ is to take into account the interaction between neighboring representative volumes. Then, in the inhomogeneity Ω_K,

$$\tilde{\sigma}^O + \tilde{\sigma}_\Omega = \tilde{C}^*(\tilde{\varepsilon}^O + \tilde{\varepsilon}_D + \langle\tilde{\gamma}\rangle - \tilde{\varepsilon}_\Omega^P), \tag{3}$$

where $\tilde{\gamma}$ expresses the disturbed strain field due to the existence of the inhomogeneity and misfit by plasticity inside the representative volume; $\langle\ \ \rangle$ indicates the average over Ω_K; $\tilde{\varepsilon}_\Omega^P$ is the average plastic strain in Ω_K. Since all inhomogeneities are of the same shape with the same material properties, the average over Ω_K is identical with that over Ω.

Substitution of eqn (2) into eqn (3) yields

$$\tilde{\sigma}^O + \tilde{\sigma}_\Omega = \tilde{C}^*\{\tilde{C}^{-1}(\tilde{\sigma}^O + \tilde{\sigma}_M) + \langle\tilde{\gamma}\rangle - \Delta\tilde{\varepsilon}^P\}, \tag{4}$$

where

$$\Delta\bar{\varepsilon}^P \equiv \bar{\varepsilon}_\Omega^P - \bar{\varepsilon}_M^P. \tag{5}$$

From eqn (4), the local disturbed strain field $\bar{\gamma}$ due to plasticity is caused only by the misfit of the plastic strain $\Delta\bar{\varepsilon}^P$. It is natural because the superposition of the uniform plastic strain $(-\bar{\varepsilon}_M^P)$ over the entire body does not disturb the entire stress field. Applying the equivalent inclusion method, we have the equivalency condition in one inhomogeneity as

$$\bar{\sigma}^O + \bar{\sigma}_\Omega = \bar{C}^*\{\bar{C}^{-1}(\bar{\sigma}^O + \bar{\sigma}_M) + \langle\bar{\gamma}\rangle - \Delta\bar{\varepsilon}^P\} \tag{6a}$$

$$= \bar{C}\{\bar{C}^{-1}(\bar{\sigma}^O + \bar{\sigma}_M) + \langle\bar{\gamma}\rangle - (\Delta\bar{\varepsilon}^P + \langle\bar{\varepsilon}^*\rangle)\}, \tag{6b}$$

where $\bar{\varepsilon}^*$ denotes the eigenstrain. Then the local disturbance $\bar{\gamma}$ is given by (see Mura, 1982)

$$\gamma_{ij}(\bar{x}) = -\int_{\Omega_K} C_{klmn}\{\Delta\varepsilon_{mn}^P(\bar{\xi}) + \varepsilon_{mn}^*(\bar{\xi})\}\tfrac{1}{2}\{G_{ik,lj}(\bar{x}-\bar{\xi}) + G_{jk,li}(\bar{x}-\bar{\xi})\}\,\mathrm{d}\bar{\xi}. \tag{7}$$

When the inhomogeneity has the ellipsoidal shape, integration of eqn (7) over Ω_K leads to

$$\langle\bar{\gamma}\rangle = \bar{S}(\Delta\bar{\varepsilon}^P + \langle\bar{\varepsilon}^*\rangle), \tag{8}$$

where $\bar{S}$ becomes constant in terms of dimensions of inhomogeneity and Poisson's ratio of the matrix, and is called Eshelby's tensor (Eshelby, 1957).

Substitution of eqn (8) into eqn (6b) yields

$$\bar{\sigma}_\Omega = \bar{\sigma}_M + \bar{C}(\bar{S} - \bar{I})(\Delta\bar{\varepsilon}^P + \langle\bar{\varepsilon}^*\rangle), \tag{9}$$

where $\bar{I}$ is the identity tensor. From eqns (1) and (9), we have

$$\bar{\sigma}_M = -f\bar{C}(\bar{S} - \bar{I})(\Delta\bar{\varepsilon}^P + \langle\bar{\varepsilon}^*\rangle). \tag{10}$$

Putting eqn (10) back into eqn (9), we obtain

$$\bar{\sigma}_\Omega = (1-f)\bar{C}(\bar{S} - \bar{I})(\Delta\bar{\varepsilon}^P + \langle\bar{\varepsilon}^*\rangle). \tag{11}$$

On the other hand, the equivalency condition, eqn (6), with eqn (8) results in

$$\Delta\bar{\varepsilon}^P + \langle\bar{\varepsilon}^*\rangle = [\bar{C} - (\bar{C} - \bar{C}^*)\{\bar{S} - f(\bar{S} - \bar{I})\}]^{-1}\{(\bar{C} - \bar{C}^*)\bar{C}^{-1}\bar{\sigma}^O + \bar{C}^*\Delta\bar{\varepsilon}^P\}. \tag{12}$$

Therefore the average stress fields in the matrix and inhomogeneity are obtained from eqns (10)–(12) as follows:

$$\bar{\sigma}^O + \bar{\sigma}_M = \bar{\sigma}^O - f\bar{C}(\bar{S} - \bar{I})[\bar{C} - (\bar{C} - \bar{C}^*)\{\bar{S} - f(\bar{S} - \bar{I})\}]^{-1}\{(\bar{C} - \bar{C}^*)\bar{C}^{-1}\bar{\sigma}^O + \bar{C}^*\Delta\bar{\varepsilon}^P\},$$

$$\bar{\sigma}^O + \bar{\sigma}_\Omega = \bar{\sigma}^O + (1-f)\bar{C}(\bar{S} - \bar{I})[\bar{C} - (\bar{C} - \bar{C}^*)\{\bar{S} - f(\bar{S} - \bar{I})\}]^{-1} \times\{(\bar{C} - \bar{C}^*)\bar{C}^{-1}\bar{\sigma}^O + \bar{C}^*\Delta\bar{\varepsilon}^P\}. \tag{13}$$

In order to calculate the average strain field, it is convenient to express its elastic part. From eqns (2) and (4), we can write

$$\bar{\varepsilon}_M^e = \bar{C}^{-1}(\bar{\sigma}^O + \bar{\sigma}_M),$$

$$\bar{\varepsilon}_\Omega^e = \bar{C}^{-1}(\bar{\sigma}^O + \bar{\sigma}_M) + \langle\bar{\gamma}\rangle - \Delta\bar{\varepsilon}^P, \tag{14}$$

where $\bar{\varepsilon}_M^e$ and $\bar{\varepsilon}_\Omega^e$ are the elastic part of the average strain fields in the matrix and inhomogeneity, respectively. Let $\bar{\varepsilon}$ denote the overall average strain, and $\bar{\varepsilon}^P$ the average plastic strain defined by

$$\bar{\varepsilon}^p \equiv (1-f)\bar{\varepsilon}^p_M + f\bar{\varepsilon}^p_\Omega. \tag{15}$$

Then the overall average elastic strain can be given by

$$\begin{aligned}\bar{\varepsilon}^e &\equiv \bar{\varepsilon} - \bar{\varepsilon}^p \\ &= (1-f)\bar{\varepsilon}^e_M + f\bar{\varepsilon}^e_\Omega.\end{aligned} \tag{16}$$

Considering eqns (8) and (12), and substituting eqn (14) into eqn (16), we finally obtain the overall stress–strain relation as

$$\begin{aligned}\bar{\varepsilon} - \bar{\varepsilon}^p = [\tilde{C} - (\tilde{C} - \tilde{C}^*)\{\tilde{S} - f(\tilde{S} - \tilde{I})\}]^{-1}[\{\tilde{C} - (1-f)(\tilde{C} - \tilde{C}^*)\tilde{S}\}\tilde{C}^{-1}\bar{\sigma}^O \\ + f(1-f)(\tilde{C} - \tilde{C}^*)(\tilde{S} - \tilde{I})\Delta\bar{\varepsilon}^p].\end{aligned} \tag{17}$$

3. AN EXAMPLE: SPHERICAL INHOMOGENEITY IN AN ISOTROPIC BODY

3.1. *General formulation*

As an example, let the shape of inhomogeneities be spherical. The Eshelby tensor for a spherical inhomogeneity can be decomposed into two parts as

$$\tilde{S} = \alpha\tilde{I}^1 + \beta\tilde{I}^2,$$

where the $\tilde{I}^1$ and $\tilde{I}^2$ components correspond to volumetric and shear deformation, respectively, and

$$I^1_{ijkl} = \tfrac{1}{3}\delta_{ij}\delta_{kl}, \quad \tilde{I}^2 = \tilde{I} - \tilde{I}^1,$$

in which δ_{ij} indicates the Kronecker delta. Henceforth we express such a tensor as

$$\tilde{S} = (\alpha, \beta), \tag{18}$$

where

$$\alpha \equiv \frac{1+\nu}{3(1-\nu)}, \quad \beta \equiv \frac{2(4-5\nu)}{15(1-\nu)}, \tag{19}$$

and ν is Poisson's ratio of the matrix material.

Let μ and λ denote the Lamé constants of the matrix material, and μ^* and λ^* be the corresponding constants of the inhomogeneity. Then

$$\tilde{C} = (3\kappa, 2\mu), \quad \tilde{C}^* = (3\kappa^*, 2\mu^*), \tag{20}$$

where κ is the bulk modulus defined by $(\lambda + 2\mu/3)$. Substitution of eqns (19) and (20) into eqn (13) results in

$$\bar{\sigma}^O + \bar{\sigma}_M = \bar{\sigma}^O - f\left[\frac{(\alpha-1)(\kappa-\kappa^*)}{A}, \frac{(\beta-1)(\mu-\mu^*)}{B}\right]\bar{\sigma}^O - f\left[\frac{3(\alpha-1)\kappa\kappa^*}{A}, \frac{2(\beta-1)\mu\mu^*}{B}\right]\Delta\bar{\varepsilon}^p,$$

$$\begin{aligned}\bar{\sigma}^O + \bar{\sigma}_\Omega = \bar{\sigma}^O + (1-f)\left[\frac{(\alpha-1)(\kappa-\kappa^*)}{A}, \frac{(\beta-1)(\mu-\mu^*)}{B}\right]\bar{\sigma}^O \\ + (1-f)\left[\frac{3(\alpha-1)\kappa\kappa^*}{A}, \frac{2(\beta-1)\mu\mu^*}{B}\right]\Delta\bar{\varepsilon}^p,\end{aligned} \tag{21}$$

where

$$A \equiv \kappa-\{\alpha-f(\alpha-1)\}(\kappa-\kappa^*), \quad B \equiv \mu-\{\beta-f(\beta-1)\}(\mu-\mu^*). \tag{22}$$

Similarly eqn (17) becomes

$$\bar{\varepsilon}-\bar{\varepsilon}^P = \left[\frac{\kappa-(1-f)(\kappa-\kappa^*)\alpha}{3\kappa A}, \frac{\mu-(1-f)(\mu-\mu^*)\beta}{2\mu B}\right]\bar{\sigma}^O + f(1-f)\left[\frac{(\alpha-1)(\kappa-\kappa^*)}{A}, \frac{(\beta-1)(\mu-\mu^*)}{B}\right]\Delta\bar{\varepsilon}^P. \tag{23}$$

3.2. *Pure shear*

Overall shearing behavior can be easily estimated as the simplest case, because only one component of stress tensors is necessary. Let the body considered be subjected to pure shearing in the x_1–x_2-plane. Then the yield condition in particular becomes as simple as

$$|(\sigma^O+\sigma_M)_{12}| = \tau_Y^M, \quad |(\sigma^O+\sigma_\Omega)_{12}| = \tau_Y^\Omega \tag{24a),(24b}$$

in the matrix and inhomogeneity respectively as an average. From eqns (21) and (23) we need only the $\bar{I}^2$ components to obtain

$$(\sigma^O+\sigma_M)_{12} = \sigma_{12}^O - f\frac{(\beta-1)(\mu-\mu^*)}{B}\sigma_{12}^O - f\frac{2(\beta-1)\mu\mu^*}{B}\Delta\varepsilon_{12}^P, \tag{25a}$$

$$(\sigma^O+\sigma_\Omega)_{12} = \sigma_{12}^O + (1-f)\frac{(\beta-1)(\mu-\mu^*)}{B}\sigma_{12}^O + (1-f)\frac{2(\beta-1)\mu\mu^*}{B}\Delta\varepsilon_{12}^P, \tag{25b}$$

$$(\bar{\varepsilon}-\bar{\varepsilon}^P)_{12} = \frac{\mu-(1-f)(\mu-\mu^*)\beta}{2\mu B}\sigma_{12}^O + f(1-f)\frac{(\mu-\mu^*)(\beta-1)}{B}\Delta\varepsilon_{12}^P. \tag{25c}$$

Stage 1: both materials are elastic. When both materials are in elasticity, no plastic strain exists and eqn (25c) yields

$$\sigma_{12}^O = 2\mu\frac{1-(1-m)\{\beta-f(\beta-1)\}}{1-(1-f)(1-m)\beta}\bar{\varepsilon}_{12} = 2\mu\left\{1-\frac{f(1-m)}{1-(1-f)(1-m)\beta}\right\}\bar{\varepsilon}_{12}, \tag{26}$$

where the following non-dimensional parameter is introduced:

$$m \equiv \frac{\mu^*}{\mu}. \tag{27}$$

Equation (26) exactly coincides with the result by Mori and Wakashima (1989).

Stage 2a: if the matrix yields first. At the initial yielding, the yeild condition eqn (24a) is satisfied, and therefore putting $\Delta\varepsilon_{12}^P = 0$ into eqn (25a), we obtain the overall stress at the first yielding $(\sigma_{12}^O)_c$ as

$$(\sigma_{12}^O)_c = \left[1+\frac{f(\beta-1)(1-m)}{1-\beta(1-m)}\right]\tau_Y^M. \tag{28}$$

Since the matrix material keeps satisfying the yield condition even after this initial yielding, its local stress level remains the same as the yield stress. Therefore from eqn (25a):

$$(\sigma^O+\sigma_M)_{12} = \frac{\mu-\beta(\mu-\mu^*)}{B}\sigma^O_{12} - \frac{2f\mu\mu^*(\beta-1)}{B}\Delta\varepsilon^P_{12} = \tau^M_Y. \tag{29}$$

Eliminating $\Delta\varepsilon^P_{12}$ from eqns (25c) and (29), we obtain

$$2\mu(\bar{\varepsilon}-\bar{\varepsilon}^P)_{12} = \frac{m+(1-\beta)(1-f)(1-m)}{m[1-(1-m)\{\beta-f(\beta-1)\}]}\sigma^O_{12} - \frac{(1-f)(1-m)}{m}\tau^M_Y. \tag{30}$$

As the plastic strain exists only in the matrix, $\Delta\varepsilon^P_{12} = -(\varepsilon^P_M)_{12}$. Therefore from eqns (15) and (29):

$$\begin{aligned}2\mu\bar{\varepsilon}^P_{12} &= -2\mu(1-f)\Delta\varepsilon^P_{12}\\ &= -\frac{(1-f)\{1-\beta(1-m)\}}{mf(\beta-1)}\sigma^O_{12} + \frac{(1-f)[1-(1-m)\{\beta-f(\beta-1)\}]}{mf(\beta-1)}\tau^M_Y.\end{aligned} \tag{31}$$

From eqns (30) and (31), the overall stress–strain relation can be obtained as

$$\sigma^O_{12} = 2\mu\frac{mf(1-\beta)}{(1-\beta)+m\beta(1-f)}\bar{\varepsilon}_{12} - \frac{(1-f)\{(1-m)\beta-1\}}{(1-\beta)+m\beta(1-f)}\tau^M_Y, \tag{32}$$

which shows a bilinear relation together with eqn (26). The coefficient in the first term of the right-hand side of eqn (32) represents the average overall hardening rate after the first yielding. Similarly elimination of $\Delta\varepsilon^P_{12}$ from eqns (25b) and (29) leads to the local stress state inside an inhomogeneity as

$$(\sigma^O+\sigma_\Omega)_{12} = \frac{\sigma^O_{12}}{f} - \frac{1-f}{f}\tau^M_Y. \tag{33}$$

Stage 3: then the inhomogeneity yields. As the applied stress increases, the stress state inside inhomogeneities increases to achieve its yield condition. Then substitution of eqn (33) into eqn (24b) results in

$$(\sigma^O+\sigma_\Omega)_{12} = \frac{\sigma^O_{12}}{f} - \frac{1-f}{f}\tau^M_Y = \tau^\Omega_Y,$$

and the maximum stress $(\sigma^O_{12})_u$ can be obtained as

$$(\sigma^O_{12})_u = \bar{\tau}_Y \equiv (1-f)\tau^M_Y + f\tau^\Omega_Y. \tag{34}$$

Since $\bar{\varepsilon}^P_{12}$ becomes arbitrary at this stress level, the overall strain cannot be determined uniquely, and thus this state can be called ultimate. Namely the ultimate state is achieved when the overall stress reaches the average value of the yield stresses of the matrix and inhomogeneity $\bar{\tau}_Y$.

Stage 2b: if the inhomogeneity yields first. On the contrary, if the inhomogeneity becomes plastic first, the yield condition eqn (24b) is satisfied first. Then substituting eqn (25b) into eqn (24b), we obtain the initial yield overall stress as

$$(\sigma^O_{12})_c = \left[1 - \frac{(1-f)(\beta-1)(1-m)}{m}\right]\tau^\Omega_Y. \tag{35}$$

Then the calculations and discussions similar to those in Stage 2a lead to the following expressions:

$$2\mu(\bar{\varepsilon}-\bar{\varepsilon}^P)_{12} = \sigma^O_{12} + \frac{f(1-m)}{m}\tau^{\Omega}_Y, \tag{36}$$

and

$$2\mu\bar{\varepsilon}^P_{12} = \frac{f}{(1-\beta)(1-f)}\sigma^O_{12} - \frac{f[1-(1-m)\{\beta-f(\beta-1)\}]}{m(1-\beta)(1-f)}\tau^{\Omega}_Y. \tag{37}$$

From eqns (36) and (37), the overall hardening behavior can be obtained as

$$\sigma^O_{12} = 2\mu\frac{(1-\beta)(1-f)}{(1-\beta)(1-f)+f}\bar{\varepsilon}_{12} + \frac{f}{(1-\beta)(1-f)+f}\tau^{\Omega}_Y. \tag{38}$$

The ultimate state is then achieved at the same stress level obtained at Stage 3; i.e. the maximum stress coincides with the average yield stress as given in eqn (34).

4. DISCUSSION

In the preceding section, the simplest example is solved to show the feasibility of the present method. Although the obtained results show the average behavior, a clear phenomenological tendency can be obtained explicitly and quantitatively.

Comparison of eqn (28) with eqn (35) leads to the following relation: the matrix will yield first if

$$t \equiv \frac{\tau^{\Omega}_Y}{\tau^M_Y} > \frac{m}{1-\beta(1-m)}, \tag{39}$$

otherwise the inhomogeneity will become plastic first. In the following several numerical examples, Poisson's ratio of the matrix material is kept equal to 0.3 unless otherwise stated.

Figure 1 shows the cases when the inhomogeneity is chosen to be stiffer and stronger than the matrix: i.e. $m = 100$ and $t = 3.0$ or $t = 1.2$. Addition of such particles will increase

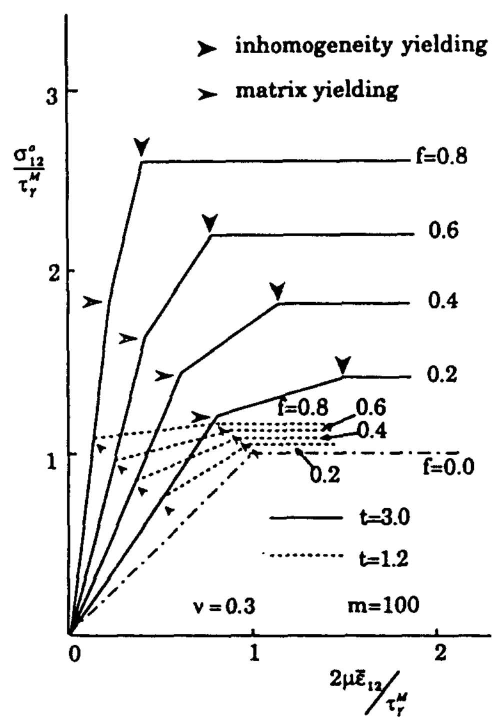

Fig. 1. Overall elastic–plastic behavior of composite materials with spherical inclusions in pure shear, $\nu = 0.3$, $m = 100$.

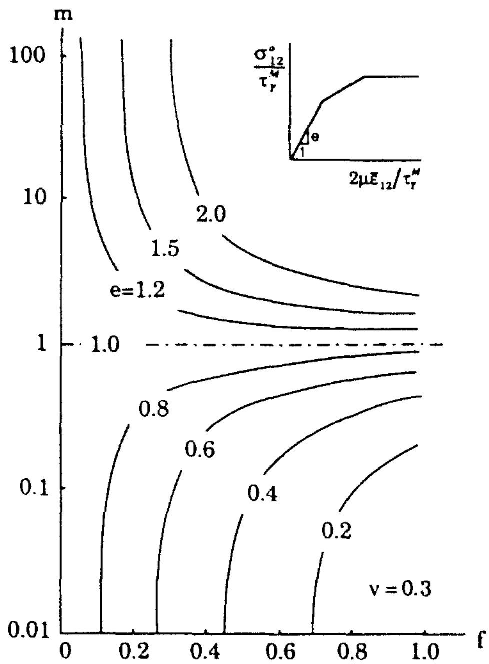

Fig. 2. Overall elastic moduli of composite materials with spherical inclusions in pure shear, $\nu = 0.3$.

not only the initial stiffness but also the ultimate strength. Furthermore the initial yield point can be also enhanced if the material parameters are chosen so that the matrix yields first, as indicated by the solid lines in the figure.

Figure 2 summarizes the overall initial elastic moduli in terms of the modulus ratio m and volume fraction f. In the case of spherical inhomogeneities, it can be shown that the predicted moduli coincide with the lower bound (see Hashin and Shtrikman, 1963) when $m > 1$, and that they are identical with the upper bound when $m < 1$. Figure 3 shows

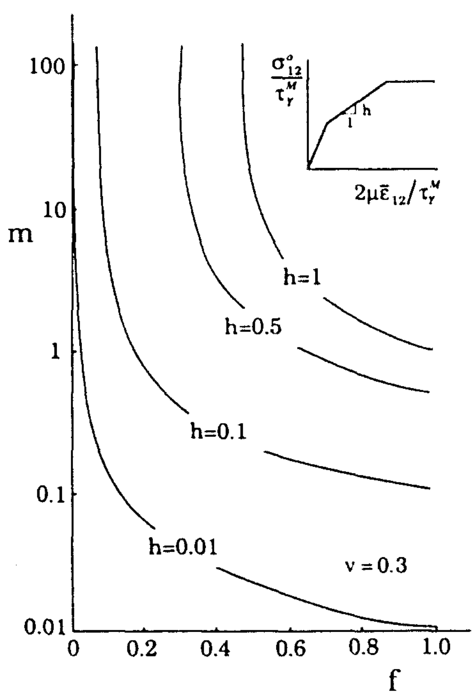

Fig. 3. Overall hardening rate of composite materials with spherical inclusions in pure shear, when yielding of the matrix occurs first $\nu = 0.3$.

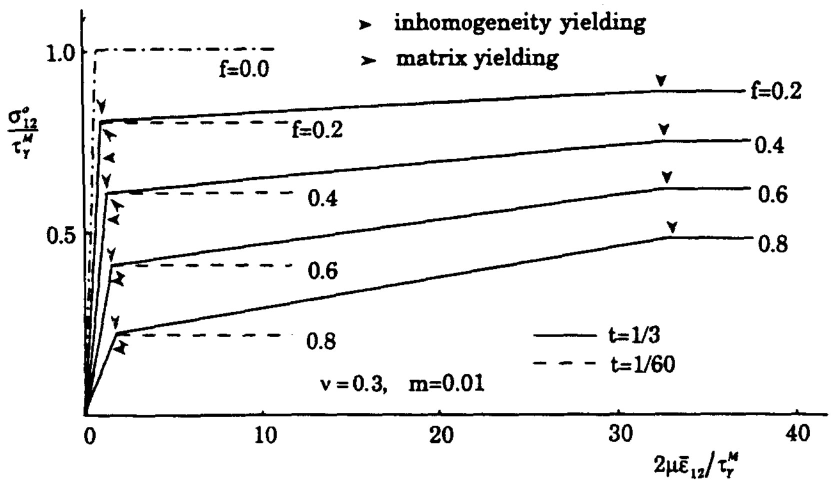

Fig. 4. Overall elastic–plastic behavior of composite materials with spherical inclusions in pure shear, $\nu = 0.3$, $m = 0.01$.

changes of the hardening rate after the first yielding; i.e. the coefficient of the first term of eqn (32), in the case when material parameters are chosen so that the matrix yields first. The lines in the figure indicate the contours of the constant rate with respect to the elastic modulus ratio m and the volume fraction f. The smaller the modulus ratio becomes, or the smaller the volume fraction is, the smaller this rate of hardening becomes.

As is clear from eqn (38), this hardening rate is independent of m in the case when the inhomogeneity yields first. The tendency is opposite to those in Fig. 3, and the larger the volume fraction is, the smaller the hardening rate becomes, as can be observed from the dashed lines in Fig. 1.

On the other hand, if softer and weaker materials are chosen for inhomogeneities, the overall material shows a more ductile property, as shown in Fig. 4: i.e. $m = 0.01$ and $t = 1/3$ or $1/60$. The ultimate strength at which both phases become plastic is reduced by addition of such inhomogeneities. However, if the matrix material is chosen so that it yields first, the softer the inhomogeneity is, the larger becomes the total deformation at the ultimate state, as indicated by the solid lines in the figure. On the contrary, if the inhomogeneity yields first, the overall material yields at a relatively lower level of the applied stress, and no improvement of ductility is observed.

In order to estimate this improvement of ductility quantitatively, we define a measure of ductility D by

$$D \equiv \frac{(\bar{\varepsilon}_{12} \text{ at } \sigma_{12}^{0} = \bar{\tau}_Y)}{\tau_Y^M/(2\mu)}, \tag{40}$$

the denominator of which expresses the yield strain of the matrix alone, and D becomes unity when no inhomogeneity exists. Using eqns (32), (34) and (38), we can express this ductility measure by

$$D = \frac{t}{m} + \frac{\beta(1-f)(1-t)}{\beta-1} \tag{41a}$$

when the matrix yields first, and

$$D = 1 + \frac{f\beta(1-t)}{1-\beta} \tag{41b}$$

when the inhomogeneity yields first. In the latter case, ductility is independent of the ratio of the elastic moduli m, as is clear from eqn (41b). The relations obtained from eqn (41) are shown in Fig. 5 as the contour lines in terms of the yield stress ratio t and elastic modulus ratio m. From eqns (34) and (39), it can be shown that the strength of composites increases by addition of inhomogeneities if $t > 1$. Therefore, the softer and stronger inhomogeneity increases both the strength and ductility of composite materials.

As for the hardening rate of composite materials, Tanaka and Mori (1970) have derived a simple formula as

$$h \equiv (\text{hardening rate}) = 2\mu \frac{fm(1-\beta)}{1-\beta(1-m)}, \tag{42a}$$

while the present method leads to eqn (32), which reads

$$h = 2\mu \frac{fm(1-\beta)}{1-\beta(1-m)-m\beta f}. \tag{42b}$$

The discrepancy lies only in the last term of the denominator of eqn (42b), and it can be negligibly small when f is very small. In fact, since eqn (42a) does not sufficiently take into account the interaction effects, it applies only when f is small and yields unacceptable results when $f = 1$. In comparison with experimental data, accuracy has been examined (see Mura, 1982). An example is a Cu–SiO_2 single crystal, where the SiO_2 particle is the spherical inhomogeneity; $\mu = 46.1$ GN m^{-2}, $\mu^* = 31.3$ GN m^{-2}, $\nu = 0.33$ and $f = 0.0052$; eqn (42a) gives $h/2\mu = 2.18 \times 10^{-3}$, while eqn (42b) yields $h/2\mu = 2.19 \times 10^{-3}$. The volume fraction is so small that the difference is small. Since eqn (42a) accounts for approximately 65% of

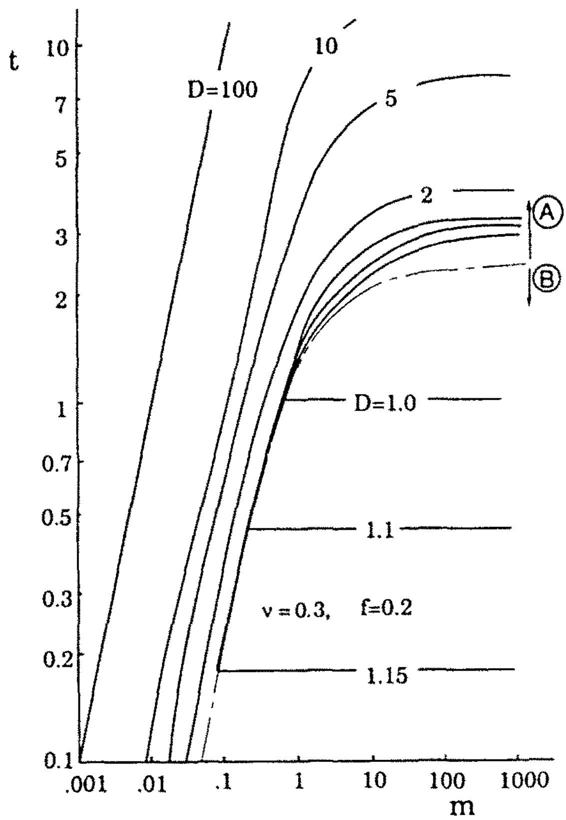

Fig. 5. Degree of ductility improvement of composite materials with spherical inclusions in pure shear, $\nu = 0.3$, $f = 0.2$. In the region indicated by B, the yielding occurs first in the inclusions, while the matrix yields first in A.

the experimentally observed hardening, the present method also gives a lower estimate of hardening.

Behavior at unloading can be also traced step-by-step similarly to the procedure above. Since both materials become elastic immediately after unloading, the overall behavior becomes the same as the initial elasticity of the composite. However, the accumulation of the residual plastic strains due to yielding shifts the next yielding point. For example, if unloading starts at a certain positive shear stress state as $\sigma^O_{12} = \sigma_a$, where $(\sigma^O_{12})_c < \sigma_a \leqslant (\sigma^O_{12})_u$, the next yielding occurs at $\sigma^O_{12} = \{\sigma_a - 2(\sigma^O_{12})_c\}$. Therefore this successive yielding occurs in the earlier stage than the virgin material by the amount of $\{\sigma_a - (\sigma^O_{12})_c\}$. Hence the overall behavior shows the Bauschinger effect of the kinematic hardening for the initial yielding, although the ultimate state is always achieved when eqn (34) is satisfied. A typical behavior in loading–unloading of composite materials is shown in Fig. 6.

As far as the Levy–Mises-type elastic–perfectly-plastic materials are concerned, the bulk modulus does not change even with the plastic deformation, because plasticity is not affected by the hydrostatic pressure component. Therefore letting $\bar{\kappa}$ denote the overall bulk modulus, we obtain, from eqn (23),

$$\frac{\bar{\kappa}}{\kappa} = 1 - \frac{f(1-k)}{1-(1-f)(1-k)\alpha}, \tag{43}$$

where k is the bulk modulus ratio defined by

$$k \equiv \frac{\kappa^*}{\kappa}. \tag{44}$$

Since this bulk modulus remains constant at any stage of deformation, $\bar{\kappa}$ can be considered as the instantaneous tangent bulk modulus. On the other hand, the tangent shear coefficient μ_t changes with deformation as is given in eqns (26), (32) and (37). In the elastic region, $\bar{\kappa}/\kappa$ shows relations similar to those in Fig. 2 for μ_t/μ, as is apparent from the comparison with eqn (26).

When the shape of isotropic inhomogeneities in an isotropic material is spherical, the overall instantaneous tangent modulus also shows an isotropy. Therefore the tangent Young's modulus and Poisson ratio can be calculated from eqns (26), (32), (37) and (43). Since $\bar{\kappa}$ remains constant and $\mu_t \to 0$ when both phases become plastic, the tangent Poisson ratio becomes 1/2. This reflects the incompressibility of the Levy–Mises-type plasticity.

A uniaxial loading condition can be also applied in the same manner as that in pure shear. In this case, the incompressibility of the plastic deformation and the axisymmetry of

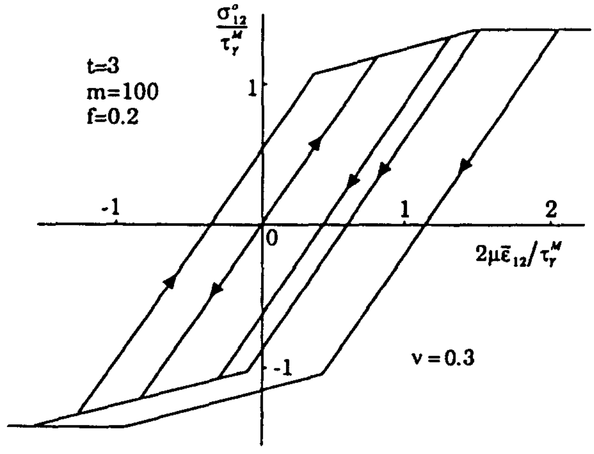

Fig. 6. Typical behavior of loading–unloading of composite materials with spherical inclusions in pure shear.

the loading condition leads to simple relations as $(\varepsilon_n^p)_{22} = (\varepsilon_n^p)_{33} = -1/2(\varepsilon_n^p)_{11}$, $(n = M, \Omega)$. If the von Mises yield condition is employed, it can be easily shown that the initial yielding occurs at $\sigma_{11}^O = (\sigma_{11}^O)_c \equiv \sqrt{3}(\sigma_{12}^O)_c$, where $(\sigma_{12}^O)_c$ is given in eqn (28) or (35). Similarly the ultimate stress is calculated as $(\sigma_{11}^O)_u = \sqrt{3}\bar{\tau}_Y$. The instantaneous tangent Young's modulus E_t is directly obtained from eqn (23), and the results are almost the same as those in Fig. 2 in the elastic range. The hardening rates of Young's modulus are obtained as

$$\frac{E}{E_t} = \frac{1}{3}\left[(1-2\nu)\left\{1+\frac{f}{\alpha-1}X\right\}+2(1+\nu)\left\{1+\frac{f}{\beta-1}Y\right\}-\frac{2(1+\nu)(1-f)(1-fY)^2}{fZ}\right], \tag{45a}$$

if the matrix yields first, and

$$\frac{E}{E_t} = \frac{1}{3}\left[(1-2\nu)\left\{1+\frac{f}{\alpha-1}X\right\}+2(1+\nu)\left\{1+\frac{f}{\beta-1}Y\right\}-\frac{2f(1+\nu)\{1+(1-f)Y\}^2}{(1-f)Z}\right], \tag{45b}$$

if the inhomogeneity becomes plastic first, where

$$X \equiv \frac{(\alpha-1)(\kappa-\kappa^*)}{A}, \quad Y \equiv \frac{(\beta-1)(\mu-\mu^*)}{B}, \quad Z \equiv \frac{(\beta-1)\mu^*}{B}, \tag{46}$$

and A and B are defined in eqn (22). Equation (45a) shows the relations quite similar to those in Fig. 3, when $m = k$.

5. CONCLUSION

A simple extension of the method to estimate elastic moduli of composites is proposed for an elastic–plastic composite material. Obtained overall behavior shows a simple bilinear property in pure shear, and the Bauschinger effect can be observed. A quantitative evaluation of ductility improvement has also been carried out.

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APPENDIX: CASE WHEN SURFACE DISPLACEMENT IS PRESCRIBED

Consider that the surface displacement or the far-field strain $\bar{\varepsilon}^O$ is prescribed instead of the surface traction treated in Section 2. Let $\bar{\varepsilon}_M$ and $\bar{\varepsilon}_\Omega$ denote the disturbed strain fields of the matrix and inhomogeneity respectively. Then these disturbances must satisfy

$$(1-f)\bar{\varepsilon}_M + f\bar{\varepsilon}_\Omega = \bar{0}. \tag{A1}$$

Let $\bar{\sigma}_M$ and $\bar{\sigma}_\Omega$ denote the local stress fields in the matrix and inhomogeneity, respectively. Then these stresses can be expressed as

$$\bar{\sigma}_M = \bar{C}(\bar{\varepsilon}^O + \bar{\varepsilon}_M - \bar{\varepsilon}^p_M), \quad \text{in} \quad (V-\Omega_k), \tag{A2}$$

$$\bar{\sigma}_\Omega = \bar{C}^*(\bar{\varepsilon}^O + \bar{\varepsilon}_\Omega - \bar{\varepsilon}^p_\Omega), \quad \text{in} \quad \Omega_k. \tag{A3}$$

Comparison of eqns (A2) and (A3) leads to the following expression, and the equivalency condition can be also written in the following form:

$$\bar{\sigma}_\Omega = \bar{C}^*\{(\bar{\varepsilon}^O + \bar{\varepsilon}_M - \bar{\varepsilon}^p_M) + (\bar{\varepsilon}_\Omega - \bar{\varepsilon}_M) - \Delta\bar{\varepsilon}^p\} \tag{A4a}$$

$$= \bar{C}\{(\bar{\varepsilon}^O + \bar{\varepsilon}_M - \bar{\varepsilon}^p_M) + (\bar{\varepsilon}_\Omega - \bar{\varepsilon}_M) - (\Delta\bar{\varepsilon}^p + \bar{\varepsilon}^*)\}. \tag{A4b}$$

From eqn (A4), $(\bar{\varepsilon}_\Omega - \bar{\varepsilon}_M)$ is considered as a disturbed strain component due to the existence of inhomogeneity and plastic deformation. Therefore, replacing $\langle\bar{\gamma}\rangle$ in eqn (8) by $(\bar{\varepsilon}_\Omega - \bar{\varepsilon}_M)$, we have

$$\bar{\varepsilon}_\Omega - \bar{\varepsilon}_M = \bar{S}(\Delta\bar{\varepsilon}^p + \bar{\varepsilon}^*). \tag{A5}$$

Eliminating $\bar{\varepsilon}_\Omega$ from eqns (A1) and (A5), we obtain

$$\bar{\varepsilon}_M = -f\bar{S}(\Delta\bar{\varepsilon}^p + \bar{\varepsilon}^*). \tag{A6}$$

Substitution of eqn (A5) into eqn (A4b) results in an alternative expression for the local stress in the inhomogeneity as

$$\bar{\sigma}_\Omega = \bar{C}\{(\bar{\varepsilon}^O + \bar{\varepsilon}_M - \bar{\varepsilon}^p_M) + (\bar{S} - \bar{I})(\Delta\bar{\varepsilon}^p + \bar{\varepsilon}^*)\}. \tag{A7}$$

Using eqns (A2) and (A7), we can define the overall stress by the average stress as

$$\begin{aligned}\bar{\sigma} &= (1-f)\bar{\sigma}_M + f\bar{\sigma}_\Omega \\ &= \bar{C}(\bar{\varepsilon}^O + \bar{\varepsilon}_M - \bar{\varepsilon}^p_M) + f\bar{C}(\bar{S} - \bar{I})(\Delta\bar{\varepsilon}^p + \bar{\varepsilon}^*),\end{aligned}$$

and substituting eqn (A6) into it, we obtain

$$\bar{\sigma} = \bar{C}(\bar{\varepsilon}^O - \bar{\varepsilon}^p_M) - f\bar{C}(\Delta\bar{\varepsilon}^p + \bar{\varepsilon}^*). \tag{A8}$$

On the other hand, the equivalency condition eqn (A4) can be rewritten by substitution of eqns (A5) and (A6) to obtain

$$(\Delta\bar{\varepsilon}^p + \bar{\varepsilon}^*) = [\bar{C} - (1-f)(\bar{C} - \bar{C}^*)\bar{S}]^{-1}\{(\bar{C} - \bar{C}^*)\bar{\varepsilon}^O - \bar{C}\bar{\varepsilon}^p_M + \bar{C}^*\bar{\varepsilon}^p_\Omega\}. \tag{A9}$$

Simple manipulation after substitution of eqn (A9) into eqn (A8) leads to the following expression for the overall stress field:

$$\bar{\sigma} = \{\bar{C} - f\bar{C}[\bar{C} - (1-f)(\bar{C} - \bar{C}^*)\bar{S}]^{-1}(\bar{C} - \bar{C}^*)\}(\bar{\varepsilon}^O - \bar{\varepsilon}^p) + f(1-f)\bar{C}[\bar{C} - (1-f)(\bar{C} - \bar{C}^*)\bar{S}]^{-1} \times (\bar{C} - \bar{C}^*)(\bar{I} - \bar{S})\Delta\bar{\varepsilon}^p,$$

or inversely,

$$\bar{\varepsilon}^O - \bar{\varepsilon}^p = [\bar{C} - (\bar{C} - \bar{C}^*)\{\bar{S} - f(\bar{S} - \bar{I})\}]^{-1}[\{\bar{C} - (1-f)(\bar{C} - \bar{C}^*)\bar{S}\}\bar{C}^{-1}\bar{\sigma} + f(1-f)(\bar{C} - \bar{C}^*)(\bar{S} - \bar{I})\Delta\bar{\varepsilon}^p]. \tag{A10}$$

Equation (A10) defines $\bar{\sigma}$ for given $\bar{\varepsilon}^O$, but is exactly identical with eqn (17), which defines $\bar{\varepsilon}$ for given $\bar{\sigma}^O$.

Furthermore, several steps of cumbersome manipulation after substitution of eqns (A6), (A9) and (A10) into eqns (A2) and (A7) result in

$$\bar{\sigma}+\tilde{\sigma}_M = \bar{\sigma}-f\bar{C}(\bar{S}-\bar{I})[\bar{C}-(\bar{C}-\bar{C}^*)\{\bar{S}-f(\bar{S}-\bar{I})\}]^{-1}\{(\bar{C}-\bar{C}^*)\bar{C}^{-1}\bar{\sigma}+\bar{C}^*\Delta\bar{\varepsilon}^p\},$$
$$\bar{\sigma}+\tilde{\sigma}_\Omega = \bar{\sigma}+(1-f)\bar{C}(\bar{S}-\bar{I})[\bar{C}-(\bar{C}-\bar{C}^*)\{\bar{S}-f(\bar{S}-\bar{I})\}]^{-1}\{(\bar{C}-\bar{C}^*)\bar{C}^{-1}\bar{\sigma}+\bar{C}^*\Delta\bar{\varepsilon}^p\}, \quad \text{(A11)}$$

which coincide with eqn (13).